

# The quantum bit from relativity of simultaneity on an interferometer

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(Dated: September 30, 2015)

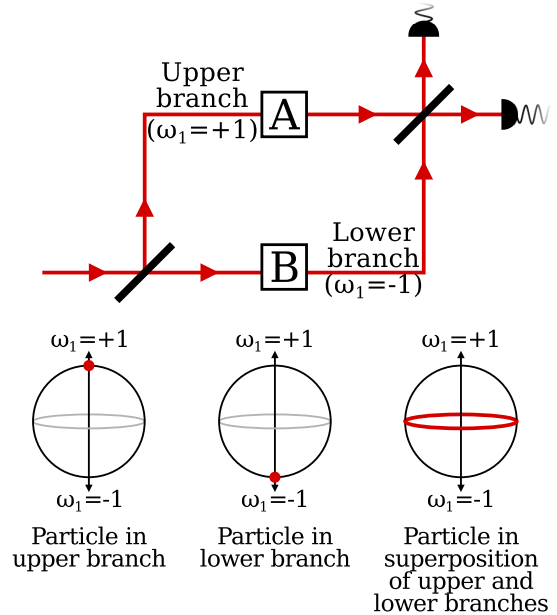
Motivated by recent experimental tests, we analyze whether relativistic spacetime could in principle allow for interference patterns more general than those predicted by quantum theory. We assume that relativity of simultaneity holds, in that the order of local transformations applied on individual arms of an interferometer cannot affect any detector click probabilities. We show that within a wide range of possible alternatives to quantum theory, this singles out the quantum Bloch sphere and thus the three degrees of freedom of a quantum bit. This has consequences for experimental tests for post-quantum interference that are currently performed in the lab, if one is to avoid their behaviour only ever replicating the quantum results.

Questions of locality have historically been used to probe the validity of quantum mechanics. A famous example is given by the EPR paradox, the subsequent discussions and experimental tests of which contrast the predictions of quantum theory against classical theory [1–5]. A relatively new development is to test quantum theory against even more general conceivable non-classical theories (e.g. [6–17]).

In this paper, we show that relativity of simultaneity [18] places significant constraints on which post-quantum theories are possible. We use the principle to derive predictions of quantum theory from within a wide range of generalisations thereof. This sheds light on the relation between quantum mechanics and spacetime, which is the major object of study in quantum gravity [19]: it suggests that the relativistic structure of spacetime itself enforces that important parts of physics are described by the standard rules of quantum theory.

Furthermore, our results are particularly relevant for a class of experiments currently being performed [20–24], which test the validity of quantum mechanics in specific interferometric setups. One proposal [23], due to Peres [20], is designed to discriminate between ordinary quantum theory and alternative theories with more than three complementary measurements. Here we evaluate what reasonable alternatives one might hope to see in these experiments if one relies on the validity of relativity, but drops other less crucial physical assumptions.

**Relativity of simultaneity on interferometers.**— Relativity of simultaneity allows for moving observers to disagree about the order in which space-like separated events happen. In particular, this principle dictates that the ordering of two space-like separated operations has no impact at all on the relative click probabilities of two detectors (labelled “1” and “2”) that are relative to each other at rest. This is because the actually observed sequence of clicks like “1211122...”, encoding the probabil-



**FIG. 1: The Mach-Zehnder interferometer (MZI).** Upper diagram shows a MZI, in which a single particle travelling from the bottom left passes through a beam-splitter and enters into a superposition of the two spatially disjoint paths. The paths recombine at the second beam splitter and the final particle position is measured by a click in a detector. Two space-like separated events A and B may alter the state of the particle in transit, and hence also the detector click statistics. The lower diagram shows some possible quantum (or beyond-quantum) states for a traversing particle.

ities as frequencies of outcomes, is an objective element of reality which must be reference-frame independent.

As a concrete example, let us look at the Mach-Zehnder interferometer (MZI) as depicted in Figure 1. A particle

(such as a photon) passes through a beamsplitter, travels through the arms, and at the end causes one of the two detectors to click. The click probability depends on the local physical conditions (say, presence of phase plates) in the two branches, with the space-like separated events A and B labelling the moment of passage through the phase plate in question. Each phase plate corresponds to some transform of the state,  $T_A$  and  $T_B$  respectively, such that the state of the system either undergoes a transformation  $T_A$  then  $T_B$  or  $T_B$  then  $T_A$ , depending on the frame of reference. Hence, relativity enforces that the probabilities of detector clicks in an MZI do not depend on the ordering of operations  $T_A$  and  $T_B$ . Quantum theory respects this requirement.

**A natural generalization of quantum theory.—**

We can, as in Figure 1, describe the state of a system passing through a two-arm interferometer with a qubit state space, which can be represented as the Bloch sphere. Phase plates are normally represented as phase gates  $U_A = \exp(i\phi_A)|0\rangle\langle 0| + |1\rangle\langle 1|$  and similarly for  $U_B$ , such that  $U_A U_B = U_B U_A$ . We will show that the specific form of these operations (and the state space they act on) is not a mathematical accident, but follows essentially from relativity of simultaneity.

We will consider a wider framework than quantum theory. There has recently been a wave of research results deriving the formalism of quantum theory from simple physical principles [8–13, 25, 26]. In most of these approaches, the first step is to prove that a two-level system is described by a ball state space; simple assumptions on the information-theoretic behaviour of a generalised bit lead to a natural generalization of the three-dimensional Bloch ball: the  $d$ -dimensional Bloch ball. These ball state spaces also feature in experimental proposals like Peres’ [20], and will thus be our subject of study here.

Let us formalise a generalised setup for a system allowing for  $d$  such complementary measurements. The state  $\omega$  of such a two-level system is an element of the  $d$ -dimensional Euclidean unit ball  $B^d \equiv \{x \in \mathbb{R}^d \mid |x| \leq 1\}$ . States  $\omega$  on the surface of the ball, i.e. with  $|\omega| = 1$ , are called *pure*, all others *mixed*. Two-outcome measurements are described by vectors  $e \in \mathbb{R}^d$  with  $|e| = 1$ ; the probability of the first outcome, if measured on state  $\omega \in B^d$ , is  $(1 + e \cdot \omega)/2$ , and that of the second outcome is  $(1 - e \cdot \omega)/2$ . Transformations which map states to states are given by  $d \times d$  orthogonal matrices  $R$  acting on  $\omega$ . They are *reversible* because by applying  $R^{-1} = R^T$ , the effect of  $R$  can be undone. In general, one has a compact group  $\mathcal{G} \subseteq O(d)$  that describes the set of all physically possible reversible transformations on the states. For simplicity, we will in this paper disregard transformations that are not reversible.

For  $d = 1$ , the unit ball becomes a line segment, and we recover a classical bit.  $\omega = +1$  and  $\omega' = -1$  are the two distinct configurations of a classical spin, and the values in between correspond to probabilistic mixtures. The only non-trivial reversible transformation is the bit flip  $R = -1$ , and thus taken together with the trivial

transformation  $1$ , we see  $\mathcal{G} = \mathbb{Z}_2$ .

If  $d = 3$ , we recover the two-level systems of standard complex quantum theory: Every complex  $2 \times 2$  density matrix  $\rho$  is in one-to-one correspondence with an element  $\omega_\rho = (\omega_\rho^1, \omega_\rho^2, \omega_\rho^3)$  of the Bloch ball  $B^3$  via  $\rho = (1 + \sum_{i=1}^3 \omega_\rho^i \sigma_i)/2$ , where  $\sigma_i$  are the Pauli matrices. In this representation, the unitary transformations  $\rho \mapsto U\rho U^\dagger$  with  $U \in \text{SU}(2)$  are rotations:  $\omega_\rho \mapsto R_U \omega_\rho$ , where  $R_U$  is a suitable element of  $\text{SO}(3)$ . The probability of outcome  $+1$  of a projective measurement with projector  $P = |\psi\rangle\langle\psi|$  can be written  $\text{tr}(P\rho) = (1 + \omega_P \cdot \omega_\rho)/2$ , where  $|\omega_P| = 1$ . In quantum theory, it is impossible to implement the “universal NOT” map  $R = -1$ , even though it is a symmetry of the Bloch ball (this corresponds to the transposition of the density matrix, which violates complete positivity). Thus, the compact group describing the physically possible reversible transformations on a qubit is  $\mathcal{G} = \text{SO}(3)$ .

The special case  $d = 2$  corresponds to quantum theory over the real numbers;  $d = 5$  describes a quaternionic quantum bit [27, 28], while  $d = 9$  describes an octonionic two-level system, which can be seen similarly as in the case of complex quantum theory explained above. The ball state spaces with arbitrary  $d \in \mathbb{N}$  have long been known in mathematical physics as examples of state spaces of Jordan algebras [29], and they have appeared in various places in quantum information theory [30, 31]. All these state spaces have  $N = 2$  perfectly distinguishable states and no more [8, 25], with every pair of antipodal points on the sphere (surface of the ball) describing mutually exclusive alternatives.

Among all reversible transformations, there are some that can be implemented locally on  $A$  (i.e. are accessible to Alice), and others on  $B$  (accessible to Bob). Both sets of transformations are groups, and we will call them  $\mathcal{G}_A$  and  $\mathcal{G}_B$ . While we do not know a priori what these groups are, there is a necessary condition for a transformation  $T \in \mathcal{G}$  to be in any of the two subgroups: it must respect locality. In the MZI, this means that  $T$  cannot change the probability to find the particle in one branch or the other (otherwise the particle would be “moved” from one arm to the other, which would be a nonlocal transformation).

To see what this implies, suppose that the “which-branch” measurement corresponds to  $e = (1, 0, \dots, 0)$  such that  $\omega = (1, 0, \dots, 0)$  describes a particle that is definitely in the upper branch, and  $\omega = (-1, 0, \dots, 0)$  one that is definitely in the lower branch (as in Figure 1). In order to not change the probability of any branch, we must have  $e \cdot \omega = e \cdot \omega'$ , where  $\omega' = T\omega$ . In other words,  $T$  must leave  $e$  invariant. The largest possible choice of both  $\mathcal{G}_A$  as well as  $\mathcal{G}_B$  is therefore the stabilizer subgroup of  $e$ , known in this context as the *phase group* of the “which-branch” measurement [32, 33]. If  $\mathcal{G} = \text{SO}(d)$ , then this corresponds to  $\mathcal{G}_A = \mathcal{G}_B = \text{SO}(d-1)$ . This choice also respects a property called *branch locality* [34], saying that the state of a particle fully localized on one arm is invariant under any transformation on the other arm.

### Relativity implies the complex quantum bit.—

We now ask which of the  $d$ -dimensional Bloch ball state spaces described above are consistent with the constraint placed by the principle of relativity of simultaneity.

**Theorem 1.** *Suppose that the group of reversible transformations is  $\mathcal{G} = \text{SO}(d)$ , and assume the largest possible choice of  $\mathcal{G}_A$  and  $\mathcal{G}_B$ . Then relativity of simultaneity enforces that the Bloch ball describing the which-path state space has dimension  $d \leq 3$ . Furthermore, only for  $d = 3$ , the qubit of standard complex quantum theory, will  $\mathcal{G}_A$  and  $\mathcal{G}_B$  be non-trivial.*

*Proof.* If  $d \geq 4$ , then  $\text{SO}(d-1)$  is non-Abelian, and hence it is possible for Alice and Bob to choose non-commuting transformations such that observers disagree about the output statistics. This violates relativity of simultaneity (see technical appendix for extended details). For both cases  $d = 1$  (classical bit) and  $d = 2$  (quantum bit over the real numbers), the only possible phase transformation is the identity, such that there are no operations at all which Alice and Bob can implement on the arms of the interferometer. Hence non-trivial phase transformations are only possible if  $d = 3$ , the standard quantum bit, and in this case we obtain the complex phase  $\mathcal{G}_A = \mathcal{G}_B = \text{SO}(2) = \text{U}(1)$ .  $\square$

Let us more critically examine the assumptions above which led us to  $\mathcal{G}_A = \mathcal{G}_B = \text{SO}(d-1)$  (we will relax them further below). Labelling the subgroup of transformations  $T \in \mathcal{G}$  satisfying  $Te = e$  as the phase group  $\mathcal{G}_\phi$ , we can break our assumptions into three parts:

- (i) The assumption that  $\mathcal{G}_A = \mathcal{G}_B$ ;
- (ii) the assumption that  $\mathcal{G}_A$  and  $\mathcal{G}_B$  together generate<sup>1</sup> the full phase group  $\mathcal{G}_\phi$ ;
- (iii) the assumption that the group of reversible transformations  $\mathcal{G}$  is  $\text{SO}(d)$ .

Assumption (i) is supported by the intuition that local transformations should be *relational*: it is the physical conditions in one arm *compared to those in the other arm* that lead to a phase transformation. This means in particular that every operation  $T_A$  (such as increasing the optical path length) can be compensated by another operation  $T_B = T_A^{-1}$  (e.g. increasing the optical path length by the same amount), which is only possible if  $\mathcal{G}_A$  and  $\mathcal{G}_B$  are one and the same group.

Now consider what happens if assumption (ii) does not hold. In this case, the smallest group  $\mathcal{G}_{AB}$  containing both  $\mathcal{G}_A$  and  $\mathcal{G}_B$  is a proper subgroup of all phase transformations  $\mathcal{G}_\phi$ . Such a situation appears physically implausible as it introduces an additional asymmetry into

the physical setup: some part of the device must make the arbitrary choice of subgroup. Clearly, the beamsplitter is the only conceivable part of the setup that could serve this purpose. But then, the beamsplitter would have to do much more than just to prepare a delocalised state, and to select a basis where the particle being in the upper branch corresponds to the state  $\omega = (1, 0, \dots, 0)$ —it would also have to select that subgroup. This would lead to the seemingly paradoxical possibility that there could be beamsplitters which prepare exactly the same states (which behave identically for all measurements), but that make Alice's and Bob's local phase plates behave as elements of different transformation groups (such that the local phase transformations, followed by a measurement, yield different statistics). The implausibility of this motivates assumption (ii) (see further discussion in the appendix).

Let us turn our attention to assumption (iii). Most generally, the linear<sup>2</sup> automorphism group of the  $d$ -ball is  $\text{O}(d)$  and the reversible dynamics will be this group or a subgroup thereof. However, if we assume that all dynamics is achieved by *continuous* time evolution, then the group of possible transformations  $\mathcal{G}$  must be connected. This is because time evolution starting at  $t = 0$  implements some  $G(t)$ , with  $\lim_{t \rightarrow 0} G(t) = \mathbf{1}$ . Since  $\text{O}(d)$  is not connected for  $d \geq 2$ ,  $\mathcal{G}$  must be a subgroup of the connected component at the identity, namely  $\text{SO}(d)$ . Assumption (iii) is thus equivalent to admitting the largest possible group of reversible transformations achievable by continuous time evolution.

Thus, the assumption  $\mathcal{G}_A = \mathcal{G}_B = \mathcal{G}_\phi = \text{SO}(d-1)$  is well-motivated by the symmetry of the setup. When this holds, the only  $d$ -ball consistent with relativity of simultaneity on a non-trivial interferometer is the standard complex quantum bit with  $d = 3$ .

**Interference beyond quantum theory.—** Let us now weaken these assumptions to shed light on what plausible behaviour we might encounter should Peres' interference experiment [20] not be governed by the rules of standard complex quantum theory, but still remain consistent with relativity of simultaneity.

In particular, let us relax (i) and (iii) by replacing them with the following weaker assumptions:

- (i\*)  $\mathcal{G}_A$  and  $\mathcal{G}_B$  are *isomorphic*;
- (iii\*) it is possible to map every pure state to every other by a reversible transformation.

Assumption (i\*) is now extremely well motivated by the symmetry of the setup: both arms are identical copies of each other, and should thus allow Alice and Bob isomorphic sets of local transformations.

Assumption (iii\*) is a property named *transitivity* on the surface of the Bloch ball [35], and is true in quantum

<sup>1</sup> All groups in this paper are assumed to be topologically closed. Since they are subgroups of the Lie group  $\text{O}(d)$ , they must therefore be compact Lie groups themselves (which includes the possibility that they are finite or not connected).

<sup>2</sup> Since transformations must respect statistical mixtures, which correspond to convex combinations of states, they must be linear.

theory but very well motivated in general, since reversible time evolution should be able to exhaust the set of all pure states. For a  $d$ -ball, this admits  $\mathcal{G} = \text{O}(d)$  (not connected) or  $\mathcal{G} = \text{SO}(d)$ . There are also proper subgroups  $\mathcal{G}$  of  $\text{SO}(d)$  which are still transitive on the surface of the ball, such as  $\text{SU}(d/2)$  for even  $d \geq 4$  [35]. It is possible to exhaustively list these groups [36], and the respective phase groups  $\mathcal{G}_\phi \subset \mathcal{G}$  formed by fixing an axis.

Using this, and knowledge about the maximality of closed subgroups [37] of  $\text{SO}(d)$  and automorphisms [38] of  $\text{U}(2)$ , one finds the following generalization of Theorem 1:

**Theorem 2.** *Under the weaker assumptions (i\*), (ii), and (iii\*), relativity of simultaneity allows for the following possibilities and no more:*

- $d = 2$  (the quantum bit over the real numbers), with  $\mathcal{G} = \text{O}(2)$  and  $\mathcal{G}_A = \mathcal{G}_B = \mathcal{G}_\phi = \mathbb{Z}_2$ ;
- $d = 3$  (the standard complex quantum bit), with  $\mathcal{G} = \text{SO}(3)$  and  $\mathcal{G}_A = \mathcal{G}_B = \mathcal{G}_\phi = \text{SO}(2) = \text{U}(1)$ ;
- $d = 4$ , with  $\mathcal{G} \simeq \text{U}(2)$  and  $\mathcal{G}_A = \mathcal{G}_B = \mathcal{G}_\phi = \text{SO}(2) = \text{U}(1)$ ,
- $d = 5$  (the quaternionic quantum bit), with  $\mathcal{G} = \text{SO}(5)$ ,  $\mathcal{G}_\phi = \text{SO}(4)$ ,  $\mathcal{G}_A$  the left- and  $\mathcal{G}_B$  the right-isoclinic rotations in  $\text{SO}(4)$  (or vice versa), such that both are isomorphic to  $\text{SU}(2)$  and  $\mathcal{G}_A \cap \mathcal{G}_B = \{+\mathbb{1}, -\mathbb{1}\}$ .

[Details of  $d = 4, 5$  are given in the technical appendix.]

In addition to the  $d = 3$  case of standard complex quantum theory, there are now three more exotic possibilities that are consistent with relativity of simultaneity.

The  $d = 2$  case encounters problems with continuous time evolution, since it is not connected (as discussed above).

The  $d = 4$  case is more interesting and quite unexpected. It has recently appeared in totally different context in [39]. This group will act on a three-dimensional subspace of the sphere in the same way as phase plates on a standard complex quantum bit (see appendix for details). The group of transformations that can be applied locally on the interferometer arms is isomorphic to the group of standard qubit transformations, which suggests that this theory would behave very similarly to the standard quantum bit for experiments of this kind.

For  $d = 5$ , the transformation groups  $\mathcal{G}_A$  and  $\mathcal{G}_B$  are merely isomorphic (for  $d = 2, 3, 4$ , the strong assumption (i) is obeyed such that  $\mathcal{G}_A = \mathcal{G}_B$ ). The two groups in this case can be explicitly constructed [40] by considering the quaternionic phase matrices  $U_A = \begin{pmatrix} q_A & 0 \\ 0 & 1 \end{pmatrix}$  and  $U_B = \begin{pmatrix} 1 & 0 \\ 0 & q_B \end{pmatrix}$ , with unit quaternions  $q_A, q_B \in \mathbb{H}$ ,  $|q_A| = |q_B| = 1$ . The only elements in common between the two branches are  $\{\mathbb{1}, -\mathbb{1}\}$ . Thus, there is some non-trivial interference behaviour, but unlike complex quantum theory, an arbitrary phase induced on one branch cannot in general be cancelled out by an operation on the other. This would amount to a “non-relational” physics which is completely different to the other cases, and which could be falsified by an experiment in where one incrementally changes the phase plate on one branch (altering the output statistics), and then finds a suitable transformation on the second branch that cancels out this change.

**Conclusion.**— We have shown that within a natural family of generalisations of quantum theory, the quantum case is singled out by essentially demanding relativity of simultaneity. Is quantum theory the only probabilistic theory consistent with relativistic spacetime, or more concretely is the quantum path integral rule for summing up complex phases a direct consequence of the structure of spacetime? Our results suggest this may well be the case. However, a more detailed analysis would have to look at more involved physical situations, in particular interferometers with more than two arms.

Given the results above, it seems particularly tempting to test experimentally for a  $d = 4$  Bloch ball, which has one degree of freedom more than the standard complex quantum bit, but still one less than the quaternionic quantum bit ( $d = 5$ ). While Peres’ proposal is in principle suitable to test for this, our results show that the actual experimental implementation has to be chosen very carefully: as long as the state space will be probed only by applying different, spatially separated phase plates on identical input states (which has been proposed for some setups [24]), relativity of simultaneity is likely to prevent any detectable difference in behaviour to the standard complex quantum bit.

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**Acknowledgements.**— We are grateful for illuminating discussions and correspondence with Časlav Brukner, Jon Barrett, Howard Barnum, Borivoje Dakić, Nana Liu, Felix Pollock, and Benjamin Yadin. We furthermore acknowledge financial support from the EPSRC (UK) and the John Templeton Foundation. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

## TECHNICAL APPENDIX

**Relativity of simultaneity requires phase operations to commute.**— Let us evaluate in a little more detail our main demand that the probabilities of all allowed measurements in an MZI do not depend on the ordering of operations on either branch.

Suppose we have two observers, say Rachael and Steven, who both observe the interference experiment, but are moving at relativistic speed relative to each other, as depicted in Figure 2. Then there will be a Lorentz transformation  $\Lambda$  relating Rachael’s frame of reference to Steven’s. A priori, the state  $\omega'$  that Steven sees might be different from the state  $\omega$  experienced by Rachael, so long as they also describe measurements by different vectors  $\omega'_P, \omega_P$  such that the outcome probabilities agree (i.e.  $\omega_P \cdot \omega = \omega'_P \cdot \omega'$ ).

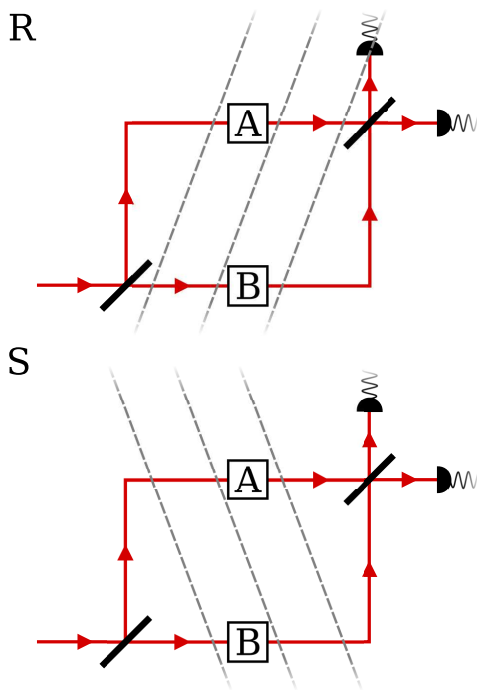


FIG. 2: **Relativity of simultaneity.** Two moving observers Rachael ( $R$ ) and Steven ( $S$ ) witness the operation of an MZI (see Figure 1). Simultaneous events in the arms of the interferometer are parallel to the dashed grey lines in the top diagram for Rachael, and to the dashed grey lines in the bottom diagram for Steven. Rachael therefore witnesses event A happening before event B; whereas Steven witnesses the opposite. If the transformations induced at A and at B do not commute, they could disagree about which detector ultimately clicks. Under relativity of simultaneity, these transformations must commute for the interference pattern to reflect an objective element of reality.

The “which branch?” freedom described by the state space is not geometric (i.e. not relative to a momentum, such as photon polarization is), but rather discriminates between two unambiguously labelled alternatives (“in Alice’s branch” vs. “in Bob’s branch”). No Lorentz boost

can turn Alice into Bob. Observers in different reference frames thus should agree on which branch is Alice’s and which is Bob’s, and when a particle travels down a branch they should also agree on which branch was traversed. As such, the transformations done by Alice and Bob that alter these Lorentz-invariant statistics must also appear the same in every reference frame.

If there is at least one pair of transformations  $T_A$  and  $T_B$  such that  $[T_A, T_B] \neq 0$ , then the order in which Alice and Bob choose to apply their transformations will have an observable effect on the output statistics of the interferometer at least for some states. This is particularly problematic for a space-like separation between Alice and Bob: Consider again the two observers Rachael and Steven, moving at relativistic speed relative to each other, as in Figure 2. Although Rachael and Steven must agree what effect either action  $T_A$  or  $T_B$  would have on the interferometer individually (from the Lorentz invariance of transformations), they could in general disagree about the order in which the two events occur [18].

In particular, let us say Rachael observes the application of  $T_A$  by Alice followed by  $T_B$  by Bob; the compound operation is then  $T_B T_A$ . When changing into Steven’s reference frame, the compound operation should be the same. However, Steven may observe instead that  $T_B$  happens before  $T_A$ , describing the compound operation by  $T_A T_B$ . This will lead to contradiction unless  $T_A T_B = T_B T_A$  in all cases; that is  $[T_A, T_B] = 0$  for all  $T_A \in \mathcal{G}_A$  and  $T_B \in \mathcal{G}_B$ .

Note the subtle way that relativity impacts this interference setup: on the one hand, the “which path” information is *not* a geometric degree of freedom that transforms in any non-trivial way under the Lorentz group. In other words, the two material interferometer arms break manifest Lorentz invariance, and all observers see the same transformations  $T_A$  and  $T_B$ . On the other hand, the local choices or applications of the transformations  $T_A$  and  $T_B$  are classical space-like separated events, which must not admit a unique time-ordering. Thus,  $T_A$  and  $T_B$  must commute.

**Further discussion of assumption (i).**— Let us first discuss a situation where assumption (i) does *not* hold, and then argue why it should hold in our two-armed interferometer. Imagine a situation where we have a single particle on four interferometer arms, and two of the arms (say,  $|1\rangle$  and  $|2\rangle$ ) are held by Alice, whereas the other two arms ( $|3\rangle$  and  $|4\rangle$ ) are held by Bob. We can think of a “which-party”-measurement, that determines whether the particle is at Alice’s or Bob’s location.

Consider a unitary  $U$  that acts non-trivially only on the subspace spanned by  $|1\rangle$  and  $|2\rangle$ . This is a phase transformation for the “which-party”-measurement. It can map  $|1\rangle$  onto an arbitrary superposition  $\alpha|1\rangle + \beta|2\rangle$ , and  $|2\rangle$  onto an arbitrary orthogonal state in the subspace, while leaving  $|3\rangle$  and  $|4\rangle$  invariant. This unitary does not alter the probability of finding the particle in Alice’s set of branches.

Clearly,  $U$  is a phase transformation that is locally

available to Alice, but not to Bob (this can be verified by arguments of *branch locality* [34], as  $U$  will have no observable effect on the output statistics for states that have no support in  $|1\rangle$  or  $|2\rangle$ ). Thus, we expect that the map  $\rho \mapsto U\rho U^\dagger$  is an element of  $\mathcal{G}_A$  (but not of  $\mathcal{G}_B$ ). There is another operational argument for this: suppose that Alice decides to either apply  $U$ , or not to apply it. After this, she determines whether the particle is found in her half of the interferometer (by performing a “which-party”-measurement). If she finds the particle in her half, then some properties of the particle will depend on whether she has applied  $U$  before or not. In other words, *applying  $U$  has a locally detectable consequence on outcomes of measurements* (conditional on finding the particle there). These consequences are observable by Alice, but not by Bob. This is the physical origin of the fact that  $U \bullet U^\dagger \in \mathcal{G}_A$  and  $U \bullet U^\dagger \notin \mathcal{G}_B$  – the mechanism that locates this transformation at  $A$ , but not at  $B$ .

In our setting, however, no localization mechanism of this kind is possible: whenever Alice or Bob perform a “which-party”-measurement (which is just a “which-path”-measurement), the state will collapse either to  $\omega = (1, 0, \dots, 0)$  or to  $\omega = (-1, 0, \dots, 0)$ , independently of any phase transformations that may have been applied before. Thus, there is no physical basis of locating some transformations  $T \in \mathcal{G}_\phi$  either at  $A$  or at  $B$ . Unless there is an additional mechanism that breaks the symmetry between Alice and Bob, we should expect to have  $\mathcal{G}_A = \mathcal{G}_B$ .

This is consistent with the findings of [33, 34], in which branch locality in general provides a mechanism for breaking the symmetry of phase operations between branches by explicitly identifying whether a particular transformation can be done on any particular branch. For two-level state spaces with an uncertainty principle (such as  $d$ -balls), it was found that branch locality does not break the symmetry between Alice and Bob. The argument above takes this a step further, by noting that *no mechanism* for breaking the symmetry between branches of an interferometer will have an observable effect on a two-level system subject to an uncertainty principle.

Note that  $\mathcal{G}_A = \mathcal{G}_B$  is a stronger statement than just saying that  $\mathcal{G}_A$  and  $\mathcal{G}_B$  are isomorphic: it says that *every action that can locally be performed by Alice can in principle also be locally performed by Bob*. In a standard quantum Mach-Zehnder interferometer, this is indeed the case: inserting a phase plate of angle  $\phi$  into Alice’s arm is completely equivalent to inserting a phase plate of angle  $(-\phi)$  into Bob’s arm.

We can also understand this property as saying that these transformations are *fully relational*: they have no locally detectable consequences whatsoever, but they alter the relation between the two interferometer arms (such as their relative optical path lengths). Thus, one would expect that either *both Alice and Bob* are able to trigger such a change of relation between the arms (by applying a suitable transformation), or neither of them are. This amounts to  $\mathcal{G}_A = \mathcal{G}_B$ .

**Further discussion of assumption (ii).**— Suppose that the smallest group that contains both  $\mathcal{G}_A$  and  $\mathcal{G}_B$ , call it  $\mathcal{G}_{AB}$ , is a proper subgroup of all phase transformations  $\mathcal{G}_\phi$ . We will restrict our discussion to the case that  $\mathcal{G}_{AB}$  is connected (and non-trivial), which is motivated by the continuity of time evolution. In this case, we can also assume that  $\mathcal{G}_\phi$  is connected, since all transformations in  $\mathcal{G}_\phi$  which are not connected to the identity will then not have any significance for the interference experiment.

First consider the case that  $\mathcal{G}_\phi$  is a simple Lie group – that is, it is connected, non-Abelian, and every closed connected normal subgroup is either  $\{1\}$  or all of  $\mathcal{G}_\phi$ . It follows that either  $\mathcal{G}_\phi$  is trivial (in which case there are no local operations on the interferometer arms whatsoever), or that  $\mathcal{G}_{AB}$  is not a normal subgroup. In the latter case, there are many “copies” of  $\mathcal{G}_{AB}$  inside of  $\mathcal{G}_\phi$ , namely  $X\mathcal{G}_{AB}X^{-1}$  for  $X \in \mathcal{G}_\phi$ . But then, one should ask *which physical mechanism selects the actual subgroup  $\mathcal{G}_{AB}$  from this infinitude of possibilities?* In other words: the proper selection of subgroup is an additional choice to be made; some element of the physical setup (i.e. the beam-splitter) has to break the symmetry and perform this choice.

Now consider the case that  $\mathcal{G}_\phi$  is *not* a simple Lie group. Considering all the possible phase groups [36] starting with a group  $\mathcal{G}$  that is transitive on the sphere, we end up with only one non-simple possibility: that  $\mathcal{G}_\phi = \text{SO}(4)$ , which is the phase group for Bloch ball dimension  $d = 5$  – that is, the quaternionic quantum bit. In this case, the argument above can be avoided if  $\mathcal{G}_{AB}$  is a non-trivial normal subgroup of  $\mathcal{G}_\phi$ . There are exactly two groups of this kind [41], both isomorphic to  $\text{SU}(2)$ , which is due to the double cover  $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$ .

Since we should expect  $\mathcal{G}_A$  and  $\mathcal{G}_B$  to be at least *isomorphic* as groups, it is easy to see that there is no way to choose them as subgroups of  $\text{SU}(2)$  such that  $[\mathcal{G}_A, \mathcal{G}_B] = 0$  and such that  $\text{SU}(2)$  is generated from the union of both groups. Thus,  $\mathcal{G}_{AB} \simeq \text{SU}(2)$  does not work, and we obtain the same conclusion as above also in this special case of the quaternionic qubit:  $\mathcal{G}_A$  and  $\mathcal{G}_B$  should generate all of  $\mathcal{G}_\phi$ .

**$d = 4$  case of Theorem 2.**— To understand how  $\mathcal{G} = \text{U}(2)$  acts on the Bloch 4-ball, note that every unitary matrix  $U \in \text{U}(n)$  can be written  $U = \text{Re } U + i \cdot \text{Im } U$ , and then the block matrix

$$G = \begin{pmatrix} \text{Re } U & \text{Im } U \\ -\text{Im } U & \text{Re } U \end{pmatrix} \quad (1)$$

is a  $2n \times 2n$  real-valued orthogonal matrix. The 4-dimensional  $\mathcal{G}$  consists of those maps for  $U \in \text{U}(2)$ . This is the same group  $\text{U}(2)$  that also acts on the pure states in standard complex quantum theory; however, the group actions are different in the way they affect the outcome probabilities of measurements. It is easy to compute the phase group preserving the vector  $\omega_1 = (1, 0, 0, 0)$ , re-

sulting in the set of maps of the form

$$G_\phi(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & \sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (2)$$

As one can see, these maps leave the  $\omega_3$ -subspace invariant, and behave in the  $(\omega_1, \omega_2, \omega_4)$ -subspace exactly in the same way as phase plates on a standard complex quantum bit.

**d = 5 case of Theorem 2.**— Suppose we relax assumption (i) that  $\mathcal{G}_A = \mathcal{G}_B$ , and instead impose assumption (i\*) that  $\mathcal{G}_A$  and  $\mathcal{G}_B$  may be distinct groups but they should still be isomorphic. It is easy to see that

$$\mathcal{G}'_A := \{G \in \mathcal{G}_\phi \mid G = XGX^{-1} \text{ for all } X \in \mathcal{G}_A\} \quad (3)$$

is a normal subgroup of  $\mathcal{G}_\phi$ , and  $\mathcal{G}_B \subseteq \mathcal{G}'_A$ , thus  $\mathcal{G}'_A$  is non-trivial. If  $\mathcal{G}'_A \subsetneq \mathcal{G}_\phi$ , then we have found a non-trivial normal subgroup of  $\mathcal{G}_\phi$ . Consider the other case that  $\mathcal{G}'_A = \mathcal{G}_\phi$ . Since  $\mathcal{G}_A \subseteq \mathcal{G}_\phi$ , this implies that  $\mathcal{G}_A$  is

Abelian. Repeating the same construction with  $A \leftrightarrow B$  interchanged, we obtain one of the following cases:

- $\mathcal{G}'_B = \mathcal{G}_\phi$  as well. But then,  $\mathcal{G}_B$  must also be Abelian, and so is the whole phase group  $\mathcal{G}_\phi$  – in this case, we are back to postulating assumption (i).
- $\mathcal{G}'_B \subsetneq \mathcal{G}_\phi$ .

Thus, we are either back with both assumptions (i) and (ii), or we have found that either  $\mathcal{G}_A$  or  $\mathcal{G}_B$  is contained in a non-trivial proper normal subgroup  $\mathcal{G}'$  of  $\mathcal{G}_\phi$ . Since  $\mathcal{G}_A$  and  $\mathcal{G}_B$  are assumed to be isomorphic and to generate all of  $\mathcal{G}_\phi$ , each of them must contain a non-trivial connected subgroup. Thus, the connected component of  $\mathcal{G}'$  at the identity,  $\mathcal{G}'_0$ , is non-trivial. Since it is a characteristic subgroup of  $\mathcal{G}'$ , and  $\mathcal{G}'$  is a normal subgroup of  $\mathcal{G}_\phi$ , the group  $\mathcal{G}'_0$  is a normal (proper non-trivial connected) subgroup of  $\mathcal{G}_\phi$  [42]. Hence  $\mathcal{G}_\phi$  is not a simple Lie group.

This admits only one possibility: that  $\mathcal{G}_\phi = \text{SO}(4)$ , i.e. we have the quaternionic quantum bit of ball dimension  $d = 5$  and transformation group  $\mathcal{G} = \text{SO}(5)$ .